Stable hypersurfaces with constant scalar curvature in Euclidean spaces

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September 11, 2009

Abstract. We obtain some nonexistence results for complete noncompact stable hyppersurfaces with nonnegative constant scalar curvature in Euclidean spaces. As a special case we prove that there is no complete noncompact strongly stable hypersurface M in $\mathbb{R}^4$ with zero scalar curvature S_2, nonzero Gauss-Kronecker curvature and finite total curvature (i.e. $\int_M |A|^3 < +\infty$).

Key words: scalar curvature, stability, index, hypersurface.

1 Introduction

In this paper we study the complete noncompact stable hypersurfaces with constant scalar curvature in Euclidean spaces. It has been proved by Cheng and Yau [CY] that any complete noncompact hypersurfaces in the Euclidean space with constant scalar curvature and nonnegative sectional curvature must be a generalized cylinder. Note that the assumption of nonnegative sectional curvature is a strong condition for hypersurfaces in the Euclidean space with zero scalar curvature since the hypersurface has to be flat in this case. Let M^n be a complete orientable Riemannian manifold and let $x : M^n \to \mathbb{R}^{n+1}$ be an isometric immersion into the Euclidean space $\mathbb{R}^{n+1}$ with constant scalar curvature. We can choose a a global unit normal vector field N and the Riemannian connections ∇ and $\widetilde{\nabla}$ of M and $\mathbb{R}^{n+1}$, respectively, are related by

$$\widetilde{\nabla}_X Y = \nabla_X Y + \langle A(X), Y\rangle N,$$

where A is the second fundamental form of the immersion, defined by

$$A(X) = -\widetilde{\nabla}_X N.$$

Let $\lambda_1, ..., \lambda_n$ be the eigenvalues of A. The *r-mean curvature* of the immersion in a point p is defined by

$$H_r = \frac{1}{\binom{n}{r}} \sum_{i_1<...<i_r} \lambda_{i_1}...\lambda_{i_r} = \frac{1}{\binom{n}{r}} S_r,$$

*The authors were partially supported by CNPq and FAPERJ, Brazil.
MSC2010 Classification: 53C42

where S_r is the r-symmetric function of the $\lambda_1, ..., \lambda_n$, $H_0 = 1$ and $H_r = 0$, for all $r \geq n+1$. For $r = 1$, $H_1 = H$ is the mean curvature of the immersion, in the case $r = 2$, H_2 is the normalized scalar curvature and for $r = n$, H_n is the Gauss-Kronecker curvature.

It is well-known that hypersurfaces with constant scalar curvature are critical points for a geometric variational problem, namely, that associated to the functional

$$\mathcal{A}_1(M) = \int_M S_1 \, dM, \tag{1}$$

under compactly supported variations that preserves volume. Let M be a hypersurface in the Euclidean space with constant scalar curvature. Following [AdCE], when the scalar curvature is zero, we say that a regular domain $D \subset M$ is *stable* if the critical point is such that $(\frac{d^2\mathcal{A}_1}{dt^2})_{t=0} \geq 0$, for all variations with compact support in D and when the scalar curvature is nonzero, we say that a regular domain $D \subset M$ is *strongly stable* if the critical point is such that $(\frac{d^2\mathcal{A}_1}{dt^2})_{t=0} \geq 0$, for all variations with compact support in D. It is natural to study the global properties of hypersurfaces in the Euclidean space with constant scalar curvature. For example we have the following open question (see 4.3 in [AdCE]).

Question 1.1 *Is there any stable complete hypersurfaces M in $\mathbb{R}^4$ with zero scalar curvature and nonzero the Gauss-Kronecker curvature?*

We have a partial answer to the question 1.1.

Theorem A. (see Theorem 3.1) *There is no complete noncompact stable hypersurface M in $\mathbb{R}^{n+1}$ with zero scalar curvature S_2 and 3-mean curvature $S_3 \neq 0$ satisfying*

$$\lim_{R \to +\infty} \frac{\int_{B_R} S_1^3}{R^2} = 0, \tag{2}$$

where B_R is the geodesic ball in M.

When $S_2 = 0$, $S_1^2 = |A|^2$ we have

Corollary B. *There is no complete noncompact stable hypersurface M in $\mathbb{R}^4$ with zero scalar curvature S_2, nonzero Gauss-Kronecker curvature and finite total curvature (i.e. $\int_M |A|^3 < +\infty$).*

We remark that Shen and Zhu (see [SZ]) proved that a complete stable minimal n-dimensional hypersurface in $\mathbb{R}^{n+1}$ with finite total curvature must be a hyperplane. The above Corollary can be seen as a similar result in dimension 3 for hypersurfaces with zero scalar curvature.

We also prove the following result for hypersurfaces with positive constant scalar curvature in Euclidean space.

Theorem C. (see Theorem 3.2) *There is no complete immersed strongly stable hypersurface $M^n \to \mathbb{R}^{n+1}$, $n \geq 3$, with positive constant scalar curvature and polynomial growth of 1-volume, that is*

$$\lim_{R \to \infty} \frac{\int_{B_R} S_1 dM}{R^n} < \infty,$$

where B_R is a geodesic ball of radius R of M^n.

As a consequence of the properties of a graph with constant scalar curvature, we have the following corollary:

Corollary D. (see Corollary 4.1) *Any entire graph on $\mathbb{R}^n$ with nonnegative constant scalar curvature must have zero scalar curvature.*

This can be compared with a result of Chern [Ch] which says any entire graph on $\mathbb{R}^n$ with constant mean curvature must be minimal. It has been be known by a result of X. Cheng in [Che] (see also [ENR]) that any complete noncompact stable hypersurface in $\mathbb{R}^{n+1}$ with constant mean curvature must be minimal if $n < 5$. It is natural to ask that any complete noncompact stable hypersurface in $\mathbb{R}^{n+1}$ with nonnegative constant scalar curvature must have zero scalar curvature.

It should be remarked that Chern[Ch] proved that there is no entire graph on $\mathbb{R}^n$ with Ricci curvature less than a negative constant. We don't know whether there exists an entire graph on $\mathbb{R}^n$ with constant negative scalar curvature.

The rest of this paper is organized as follows: we include some results and definitions which will be used in the proof of our theorems in Section 2. The proof of main results are given in Section 3 and Section 4 is an appendix in which we prove some stability properties for graphs with constant scalar curvature in the Eucildean space.

2 Some stability and index properties for hypersurfaces with $S_2 = const.$

We introduce the r'th Newton transformation, $P_r : T_pM \to T_pM$, which are defined inductively by

$$\begin{aligned} P_0 = &\ I, \\ P_r = &\ S_r I - A \circ P_{r-1},\ r \geq 1. \end{aligned}$$

The following formulas are useful in the proof (see, [Re], Lemma 2.1).

$$\operatorname{trace}(P_r) = (n-r)S_r, \tag{3}$$

$$\operatorname{trace}(A \circ P_r) = (r+1)S_{r+1}, \tag{4}$$

$$\operatorname{trace}(A^2 \circ P_r) = S_1 S_{r+1} - (r+2)S_{r+2}. \tag{5}$$

From [AdCC] we have the second variation formula for hypersurfaces in a space form of constant curvature c, $\mathbb{Q}_c^{n+1}$, with constant 2-mean curvature:

$$\frac{d^2\mathcal{A}_1}{dt^2}|_{t=0} = \int_D \langle P_1(\nabla f), \nabla f\rangle dM - \int_D (S_1S_2 - 3S_3 + c(n-1)S_1)f^2 dM, \quad \forall f \in C_c^\infty(D). \tag{6}$$

Definition 2.1 *When $S_2 = 0$ and $c = 0$, M is stable if and only if*

$$\int_M \langle P_1(\nabla f), \nabla f\rangle dM \geq -3\int_M S_3 f^2 dM, \tag{7}$$

for any $f \in C_c^\infty(M)$. One can see that if $P_1 \equiv 0$, then $S_3 = 0$ and M is stable. When $S_2 = const. \neq 0$, M is stable if and only if

$$\int_D \langle P_1(\nabla f), \nabla f\rangle dM \geq \int_D (S_1S_2 - 3S_3 + c(n-1)S_1)f^2 dM,$$

for all $f \in C_c^\infty(M)$ and $\int_M f dM = 0$. We say that M is strongly stable if and only if the above inequality holds for all $f \in C_c^\infty(M)$.

Similar to minimal hypersurface we can also define the index I for hypersurfaces with constant scalar curvature. Given a relatively compact domain $\Omega \subset M$, we denote by $\mathrm{Ind}^1(\Omega)$ the number of linearly independent normal deformations with support on Ω that decrease $\mathcal{A}_1$. The *index* of the immersion are defined as

$$\mathrm{Ind}^1(M) := \sup\{\mathrm{Ind}^1(\Omega) \mid \Omega \subset M, \quad \Omega \text{ relatively compact}\}. \tag{8}$$

M is *strongly stable* if $\mathrm{Ind}^1(M) = 0$. The following result has been known in [El].

Lemma 2.1 *Let $M^n \to \mathbb{Q}_c^{n+1}$ be a noncompact hypersurface with $S_2 = const. > 0$. If M has finite index then there exist a compact set $K \subset M$ such that $M \setminus K$ is strongly stable.*

For hypersurfaces with constant mean curvature, do Carmo and Zhou [dCZ] proved that

Theorem 2.1 *Let $x : M^n \to \overline{M}^{n+1}$ be an isometric immersion with constant mean curvature H. Assume M has subexponential volume growth and finite index. Then there exist a constant R_0 such that*

$$H \leq -\overline{Ric}_{M\setminus B_{R_0}}(N),$$

where N is a smooth normal vector field along M and $\overline{Ric}(N)$ is the Ricci curvature of $\overline{M}$ in the normal vector N.

The technique in [dCZ] was generalized by Elbert [El] to prove the following result:

Theorem 2.2 *Let $x : M^n \to \mathcal{Q}(c)^{n+1}$ be an isometric immersion with $S_2 = constant > 0$. Assume that $\mathrm{Ind}^1 M < \infty$ and that the 1-volume of M is infinite and has polynomial growth. Then c is negative and*

$$S_2 \leq -c.$$

In particular, it implies that when $c = 0$ the hypersurfaces in the above theorem must have nonpositive scalar curvature.

3 Proof of the theorems

When $S_2 = 0$ we know that $|S_1|^2 = |A|^2$. Thus, if $S_3 \neq 0$, we have that $|A|^2 > 0$. Hence $S_1 \neq 0$ and we can choose an orientation such that P_1 is semi-positive definite. Since

$$\begin{aligned} |\sqrt{P_1}A|^2 &= \mathrm{trace}(A^2 \circ P_1) \\ &= -3S_3, \end{aligned}$$

then, when $c = 0$, M is stable if

$$\int_M \langle P_1(\nabla f), \nabla f\rangle dM \geq \int_M |\sqrt{P_1}A|^2 f^2 dM, \tag{9}$$

for any $f \in C_c^\infty(M)$.

By Lemma 4.1 in [AdCC], when $S_2 = 0$, we know that $|\nabla A|^2 - |\nabla S_1|^2 \geq 0$. In the following lemma, we characterize the equality case in some special case.

Lemma 3.1 *Let $M^n (n \geq 3)$ be a non-flat connected immersed 1-minimal hypersurface in $\mathbb{R}^{n+1}$. If $|\nabla A|^2 = |\nabla S_1|^2$ holds on all nonvanishing point of $|A|$ in M, then each component of M with $|A| \neq 0$ must be a cylinder over a curve.*

Proof. Choose a frame at p so that the second fundamental form is diagonalized. From the computations in [SSY], we have $|A|^2 = \sum_i h_{ii}^2$, and

$$\begin{aligned}
\sum_{i,j,k} h_{ijk}^2 - |\nabla |A||^2 &= [(\sum_{i,j} h_{ij}^2)(\sum_{s,t,k} h_{stk}^2) - \sum_k(\sum_{i,j} h_{ij}h_{ijk})^2](\sum_{i,j} h_{ij}^2)^{-1} \\
&= \frac{1}{2}\sum_{i,j,k,s,t}(h_{ij}h_{stk} - h_{st}h_{ijk})^2|A|^{-2} \\
&= \frac{1}{2}\left[\sum_{i,k,s,t}(h_{ii}h_{stk} - h_{st}h_{iik})^2 + \sum_s h_{ss}^2(\sum_k\sum_{i\neq j} h_{ijk}^2)\right]|A|^{-2} \\
&= \frac{1}{2}\left[\sum_{i,k,s}(h_{ii}h_{ssk} - h_{ss}h_{iik})^2 + \sum_i h_{ii}^2(\sum_k\sum_{s\neq t} h_{stk}^2)\right]|A|^{-2} \\
&\qquad + \frac{1}{2}(\sum_k\sum_{i\neq j} h_{ijk}^2) \\
&= \frac{1}{2}\left[\sum_{i,k,s}(h_{ii}h_{ssk} - h_{ss}h_{iik})^2\right]|A|^{-2} + (\sum_k\sum_{i\neq j} h_{ijk}^2) \\
&= \frac{1}{2}\left[\sum_{i,k,s}(h_{ii}h_{ssk} - h_{ss}h_{iik})^2\right]|A|^{-2} + \\
&\qquad 2\sum_{i\neq j} h_{iij}^2 + \sum_{i\neq j, j\neq k, i\neq k} h_{ijk}^2 \geq 0.
\end{aligned} \tag{10}$$

It is clear that the right hand side is nonnegative and is zero if and only if all terms on the right hand side vanish.

$$\sum_{i,j,k} h_{ijk}^2 - |\nabla |A||^2 \geq 0. \tag{11}$$

Suppose $x : M \to \mathbb{R}^{n+1}$ is the 1-minimal immersion. Since M is not a hyperplane, then $|A|$ is a nonnegative continuous function which does not vanish identically. Let p be such a point such that $|A|(p) > 0$. Then $|A| > 0$ in a connected open set U containing p. The equality in (11) implies

$$\begin{aligned} h_{jji} &= 0, \text{ for all } j \neq i, \\ h_{ijk} &= 0, \text{ for all } j \neq i, j \neq k, k \neq i \\ h_{ii}h_{ssk} &= h_{ss}h_{iik}, \text{ for all } i, s, k. \end{aligned}$$

So we have $h_{jij} = 0$, for all $j \neq i$, and from the last equation we claim that at most one i such that $h_{iii} \neq 0$. Otherwise, without the loss of generality we assume $h_{111} \neq 0$, and $h_{222} \neq 0$, we have $h_{11}h_{22k} = h_{22}h_{11k}$ for all k. This implies $h_{11} = h_{22} = 0$ by choosing $k = 1, 2$. Using again the third formula we have $h_{jj}h_{111} = h_{11}h_{jj1}$ for $j = 3, \cdots, n$. Hence $h_{jj} = 0$ for all $j = 3, \cdots, n$, which contradicts to $|A| \neq 0$.

We now assume $h_{111} \neq 0$ by continuity we can also assume $h_{11} \neq 0$. From the last equation of above equation, we have $h_{11}h_{ss1} = h_{ss}h_{111}$ for $s \neq 1$. Hence $h_{ss} = 0$ for all $s \neq 1$. This implies that M is a cylinder over a curve.

□

We are now ready to prove

Theorem 3.1 *There is no complete noncompact stable hypersurfaces in R^{n+1} with $S_2 = 0$ and $S_3 \neq 0$ satisfying*

$$\lim_{R\to+\infty} \frac{\int_{B_R} S_1^3}{R^2} = 0.$$

Proof. Assume for the sake of contradiction that there were such a hypersurface M. From Lemma 3.7 in [AdCC], we have

$$L_1 S_1 = |\nabla A|^2 - |\nabla S_1|^2 + 3S_1S_3. \tag{12}$$

Since for any $\phi \in C_c^\infty(M)$,

$$\begin{aligned} \int_M \langle P_1(\nabla(\phi S_1)), \nabla(\phi S_1)\rangle dM &= \int_M \langle P_1((\nabla\phi)S_1 + \phi\nabla S_1), (\nabla\phi)S_1) + \phi\nabla S_1\rangle dM \\ &= \int_M \langle P_1(\nabla\phi), \nabla\phi\rangle S_1^2 dM + 2\int_M \langle P_1(\nabla\phi), \nabla S_1\rangle \phi S_1 dM \\ &\quad + \int_M \phi^2 \langle P_1(\nabla S_1), \nabla S_1\rangle dM, \end{aligned}$$

then using (12) we have

$$
\begin{aligned}
\int_M \phi^2 S_1(|\nabla A|^2 - |\nabla S_1|^2)dM &= \int_M (L_1 S_1 - 3S_1S_3)\phi^2 S_1 dM \\
&= -\int_M \langle P_1(\nabla S_1), \nabla(\phi^2 S_1)\rangle dM - \int_M 3S_3\phi^2 S_1^2 dM \\
&= -\int_M \phi^2 \langle P_1(\nabla S_1), \nabla S_1\rangle dM - 2\int_M \langle P_1(\nabla\phi), \nabla S_1\rangle \phi S_1 dM - \int_M 3S_3\phi^2 S_1^2 dM \\
&= -\int_M \langle P_1(\nabla(\phi S_1)), \nabla(\phi S_1)\rangle dM + \int_M \langle P_1(\nabla\phi), \nabla(\phi)\rangle S_1^2 dM - \int_M 3S_3\phi^2 S_1^2 dM \\
&\le \int_M \langle P_1(\nabla\phi), \nabla\phi\rangle S_1^2 dM \\
&\le \int_M |\nabla\phi|^2 S_1^3 dM,
\end{aligned}
$$

for any $\phi \in C_c^\infty(M)$. Here we have used the stability inequality (7) in the fifth line and use the following consequence of (3) in the last inequality:

$$
S_1|\nabla\phi|^2 \ge \langle P_1(\nabla\phi), \nabla\phi\rangle. \tag{13}
$$

We can choose ϕ as

$$
\phi(x) = \begin{cases} \frac{2R - r(x)}{R}, & \text{on } B_{2R} \setminus B_R; \\ 1, & \text{on } B_R; \\ 0, & \text{on } M \setminus B_{2R}. \end{cases}
$$

Thus from the choice of ϕ we have $S_1(|\nabla A|^2 - |\nabla S_1|^2) \equiv 0$. Therefore the elipticity of L_1 implies $L_1 S_1 = 3S_1S_3$. From Lemma 3.1, M must be a cylinder over a curve which contradicts $S_3 \neq 0$. The proof is complete.

□

The following Lemma is of some independent interest and we include here since its second part is useful in the proof of Theorem 3.2.

Lemma 3.2 *Let M be a complete immersed hypersurface in $\mathcal{Q}_c^{n+1}$ with nonnegative constant scalar curvature $S_2 > -\frac{n(n-1)}{2}c$ and $S_1 \neq 0$.*
1)If M is strongly stable outside a compact subset, then either M has finite 1-volume, or

$$
\lim_{R\to+\infty} \frac{1}{R^2}\int_{B_R} S_1 = +\infty.
$$

2)If M is strongly stable, then

$$
\lim_{R\to+\infty} \frac{1}{R^2}\int_{B_R} S_1 = +\infty.
$$

In particular M has infinite 1-volume.

Proof. We can assume that there exists a geodesic ball $B_{R_0} \subset M$ such that $M \setminus B_{R_0}$ is strongly stable. That is,

$$\int_M (S_1 S_2 - 3S_3 + c(n-1)S_1) f^2 dM \leq \int_M \langle P_1(\nabla f), \nabla f \rangle dM, \tag{14}$$

for all $f \in \mathcal{C}_c^\infty(M \setminus B_{R_0})$.

Now, since $S_2 \geq 0$, we have (see [AdCR], p. 392)

$$H_1 H_2 \geq H_3,$$

and

$$H_1 \geq H_2^{1/2}.$$

By using that $S_1 = nH_1$, $S_2 = \dfrac{n(n-1)}{2} H_2$ and $S_3 = \dfrac{n(n-1(n-2)}{6} H_3$, it follows that

$$\frac{(n-2)}{n} S_1 S_2 \geq 3S_3,$$

that is,

$$-3S_3 \geq -\frac{(n-2)}{n} S_1 S_2. \tag{15}$$

We also have that

$$\frac{S_1}{n} \geq \left(\frac{2S_2}{n(n-1)} \right)^{1/2},$$

which implies

$$S_1 \geq \left(\frac{2n}{n-1} \right)^{1/2} S_2^{1/2}. \tag{16}$$

By using inequality (15) in (14), it follows that

$$\int_M \left(S_1 S_2 - \frac{n-2}{n} S_1 S_2 + c(n-1)S_1 \right) f^2 dM \leq \int_M \langle P(\nabla f), \nabla f \rangle dM,$$

that is,

$$\int_M \left(S_2 + \frac{n(n-1)c}{2} \right) S_1 f^2 dM \leq \frac{n}{2} \int_M \langle P(\nabla f), \nabla f \rangle dM.$$

By using (13), we obtain that

$$\int_M S_1 |\nabla f|^2 dM \geq \int_M \langle P(\nabla f), \nabla f \rangle dM$$

Therefore, there exists a constant $C > 0$ such that

$$\int_M S_1 |\nabla f|^2 dM \geq C \int_M S_1 f^2 dM. \tag{17}$$

1) When M is strongly stable outside B_{R_0}. We can choose f as

$$f(x) = \begin{cases} r(x) - R_0, & \text{on } B_{R_0+1} \setminus B_{R_0}; \\ 1, & \text{on } B_{R+R_0+1} \setminus B_{R_0+1}; \\ \frac{2R+R_0+1-r(x)}{R}, & \text{on } B_{2R+R_0+1} \setminus B_{R+R_0+1}; \\ 0, & \text{on } M \setminus B_{2R+R_0+1}, \end{cases}$$

where $r(x)$ is the distance function to a fixed point. Then

$$\frac{1}{R^2} \int_{B_{2R+R_0+1} \setminus B_{R+R_0+1}} S_1 dM + \int_{B_{R_0+1} \setminus B_{R_0}} S_1 dM \geq C \int_{B_{R+R_0+1} \setminus B_{R_0+1}} S_1 dM.$$

If the 1-volume is infinite, we can choose R large such that

$$C \int_{B_{R+R_0+1} \setminus B_{R_0+1}} S_1 dM > \int_{B_{R_0+1} \setminus B_{R_0}} S_1 dM,$$

hence

$$\lim_{R \to +\infty} \frac{1}{R^2} \int_{B_{2R+R_0+1} \setminus B_{R+R_0+1}} S_1 dM = +\infty.$$

2) When M is strongly stable we can choose a simpler test function f as

$$f(x) = \begin{cases} 1, & \text{on } B_R; \\ \frac{2R-r(x)}{R}, & \text{on } B_{2R} \setminus B_R; \\ 0, & \text{on } M \setminus B_{2R}, \end{cases}$$

which implies that when $S_1 \neq 0$,

$$\lim_{R \to +\infty} \frac{1}{R^2} \int_{B_{2R}} S_1 dM = +\infty.$$

The proof is complete. □

Theorem 3.2 *There is no complete immersed strongly stable hypersurface $M^n \to \mathbb{R}^{n+1}$, $n \geq 3$, with positive constant scalar curvature and polynomial growth of 1-volume, that is*

$$\lim_{R \to \infty} \frac{\int_{B_R} S_1 dM}{R^n} < \infty,$$

where B_R is a geodesic ball of radius R of M^n.

Proof. Suppose that M is a complete immersed strongly stable hypersurface $M^n \to \mathbb{R}^{n+1}$, $n \geq 3$, with positive constant scalar curvature. From Theorem 2.2, it suffices to show that the 1-volume $\int_M S_1 dM$ is infinite which is the part (2) of Lemma 3.2.

□

4 Graphs with $S_2 = const$ in Euclidean space

In this section we include some stability properties and estimates for entire graphs on $\mathbb{R}^n$ which may be known to experts ant not easy to find a reference. Using these facts we give the proof of Corollary 4.1. Let M^n a hypersurface of $\mathbb{R}^{n+1}$ given by a graph of a function $u : \mathbb{R}^n \to \mathbb{R}$ of class $\mathcal{C}^\infty(\mathbb{R}^n)$. For such hypersurfaces we have:

Proposition 4.1 *Let M^n a graph of a function $u : \mathbb{R}^n \to \mathbb{R}$ of class $\mathcal{C}^\infty(\mathbb{R}^n)$. Then*

1. *If $S_2 = 0$ and S_1 does not change sign on M, then M^n is a stable hypersurface.*
2. *If M has $S_2 = C > 0$, then M^n is strongly stable.*

Proof. Considerer and $f : M \to \mathbb{R}$ a $\mathcal{C}^\infty$ function with compact support and let $W = \sqrt{1+|\nabla u|^2}$. In order to calculate $\langle P_1(\nabla f), \nabla f\rangle$, write $g = fW$. Thus

$$\begin{aligned}
\langle P_1(\nabla f), \nabla f\rangle &= \langle P_1(\nabla(\frac{g}{W})), \nabla(\frac{g}{W})\rangle \\
&= \langle P_1(g\nabla\frac{1}{W} + \nabla g\frac{1}{W}), g\nabla(\frac{1}{W}) + \frac{1}{W}\nabla g\rangle \\
&= \langle gP_1(\nabla\frac{1}{W}) + \frac{1}{W}P_1(\nabla g), g\nabla(\frac{1}{W}) + \frac{1}{W}\nabla g\rangle \\
&= g^2\langle P_1(\nabla\frac{1}{W}), \nabla\frac{1}{W}\rangle + \frac{g}{W}\langle P_1(\nabla\frac{1}{W}), \nabla g\rangle \\
&\quad + \frac{g}{W}\langle P_1(\nabla g), \nabla\frac{1}{W}\rangle + \frac{1}{W^2}\langle P_1(\nabla g), \nabla g\rangle.
\end{aligned}$$

By using that P_1 is selfadjoint, we have:

$$\langle P_1(\nabla f), \nabla f\rangle = g^2\langle P_1(\nabla\frac{1}{W}), \nabla\frac{1}{W}\rangle + 2\frac{g}{W}\langle P_1(\nabla\frac{1}{W}), \nabla g\rangle + \frac{1}{W^2}\langle P_1(\nabla g), \nabla g\rangle. \tag{18}$$

On the other hand, if $\{e_1, ..., e_n\}$ is a geodesic frame along M,

$$\begin{aligned}
\mathrm{div}(fgP_1(\nabla\frac{1}{W}) &= \sum_{i=1}^n \langle\nabla_{e_i}(fgP_1(\nabla\frac{1}{W})), e_i\rangle \\
&= \sum_{i=1}^n \langle fg_iP_1(\nabla\frac{1}{W}) + f_igP_1(\nabla\frac{1}{W}) + fg\nabla_{e_i}(P_1(\nabla\frac{1}{W})), e_i\rangle \\
&= \sum_{i=1}^n \{fg_i\langle P_1(\nabla\frac{1}{W}), e_i\rangle + f_ig\langle P_1(\nabla\frac{1}{W}), e_i\rangle + fg\langle\nabla_{e_i}(P_1(\nabla\frac{1}{W})), e_i\rangle\}.
\end{aligned}$$

Since $f = \dfrac{g}{W}$, we get

$$f_i = g_i\frac{1}{W} + g\left(\frac{1}{W}\right)_i,$$

that is,

$$\begin{aligned} gf_i &= gg_i\frac{1}{W} + g^2\left(\frac{1}{W}\right)_i \\ &= fg_i + g^2\left(\frac{1}{W}\right)_i \end{aligned}$$

Hence,

$$\begin{aligned} \operatorname{div}(fgP_1(\nabla\frac{1}{W}) &= \sum_{i=1}^{n}\{fg_i\langle P_1(\nabla\frac{1}{W}), e_i\rangle + (fg_i + g^2\left(\frac{1}{W}\right)_i)\langle P_1(\nabla\frac{1}{W}), e_i\rangle\} + fgL_1(\frac{1}{W}) \\ &= \sum_{i=1}^{n}\{2fg_i\langle P_1(\nabla\frac{1}{W}), e_i\rangle + g^2\left(\frac{1}{W}\right)_i\langle P_1(\nabla\frac{1}{W}), e_i\rangle\} + fgL_1(\frac{1}{W}) \\ &= 2f\langle P_1(\nabla\frac{1}{W}), \nabla g\rangle + g^2\langle P_1(\nabla\frac{1}{W}), \nabla(\frac{1}{W})\rangle + fgL_1(\frac{1}{W}) \\ &= 2\frac{g}{W}\langle\nabla\frac{1}{W}, P_1(\nabla g)\rangle + g^2\langle P_1(\nabla\frac{1}{W}), \nabla(\frac{1}{W})\rangle + f^2WL_1(\frac{1}{W}). \end{aligned}$$

Thus,

$$2\frac{g}{W}\langle\nabla\frac{1}{W}, P_1(\nabla g)\rangle = \operatorname{div}(fgP_1(\nabla\frac{1}{W})) - g^2\langle P_1(\nabla\frac{1}{W}), \nabla(\frac{1}{W})\rangle - f^2WL_1(\frac{1}{W}). \tag{19}$$

Now, by using (19) into equation (18), we get

$$\langle P_1(\nabla f), \nabla f\rangle = \operatorname{div}(fgP_1(\nabla\frac{1}{W})) - f^2WL_1(\frac{1}{W}) + \frac{1}{W^2}\langle P_1(\nabla g), \nabla g\rangle.$$

Now, the divergence theorem implies that

$$\int_M \langle P_1(\nabla f), \nabla f\rangle dM = -\int_M f^2WL_1(\frac{1}{W})dM + \int_M \frac{1}{W^2}\langle P_1(\nabla g), \nabla g\rangle dM.$$

Choose the orientation of M in such way that $S_1 \geq 0$. Since $S_1^2 - |A|^2 = 2S_2 \geq 0$, we obtain that $S_1 \geq |A|$. Thus, $\langle P_1(\nabla g), \nabla g\rangle = S_1|\nabla g|^2 - \langle A\nabla g, \nabla g\rangle \geq (S_1 - |A|)|\nabla g|^2 \geq 0$, which implies that

$$\int_M \langle P_1(\nabla f), \nabla f\rangle dM \geq -\int_M f^2WL_1(\frac{1}{W})dM. \tag{20}$$

When S_2 is constant, we will use the following formula proved by Reilly (see [Re], Proposition C).

$$L_1(\frac{1}{W}) = L_1(\langle N, e_{n+1}\rangle) + (S_1S_2 - 3S_3)\langle N, e_{n+1}\rangle = 0,$$

where N is the normal vector of M and $e_{n+1} = (0, ..., 0, \pm 1)$, according to our choice of the orientation of M.

Thus,

$$\int_M \langle P_1(\nabla f), \nabla f\rangle dM \geq -\int_M f^2WL_1(\frac{1}{W})dM = 0$$

for all function f with compact support. Hence M is stable if $S_2 = 0$ and strongly stable in the case $S_2 \neq 0$. $\square$

Remark 4.1 *We would like to remark that the operator L_1 need not to be elliptic in the above proof.*

Proposition 4.2 *Let M^n a graph of a function $u : \mathbb{R}^n \to \mathbb{R}$ of class $\mathcal{C}^\infty(\mathbb{R}^n)$, with $S_1 \geq 0$. Let B_R be a geodesic ball of radius R in M. Then*

$$\int_{B_{\theta R}} S_1 dM \leq \frac{C(n)}{1-\theta} R^n,$$

where $C(n)$ and θ are constants, with $0 < \theta < 1$. In particular, $\int_M S_1 dM$ has polynomial growth.

Proof. Let $f : M \to \mathbb{R}$ be a be a function in $\mathcal{C}_0^\infty(M)$, that is a smooth function with compact support. Observe that

$$\operatorname{div}\left(f\frac{\nabla u}{W}\right) = f\operatorname{div}\left(\frac{\nabla u}{W}\right) + \left\langle \nabla f, \frac{\nabla u}{W}\right\rangle,$$

where $W = \sqrt{1+|\nabla u|^2}$. By using the fact that S_1 is given by $S_1 = \operatorname{div}\left(\frac{\nabla u}{W}\right)$, we have that

$$\int_M f S_1 dM = \int_M f\operatorname{div}\left(\frac{\nabla u}{W}\right) dM = -\int_M \left\langle \nabla f, \frac{\nabla u}{W}\right\rangle dM. \tag{21}$$

Now, choose a family of geodesic balls B_R that exhausts M. Fix θ, with $0 < \theta < 1$ and let $f : M \to \mathbb{R}$ be a continuous function that is one on $B_{\theta R}$, zero outside B_R and linear on $B_R \setminus B_{\theta R}$. Therefore, from equation (21) we obtain

$$\int_{B_{\theta R}} S_1 dM \leq \int_{B_R} f S_1 dM \leq \int_{B_R} \left\langle \frac{\nabla u}{W}, \nabla f\right\rangle dM.$$

By using Cauchy-Schwarz inequality and the fact that $\frac{|\nabla u|}{W} \leq 1$, it follows that

$$\int_{B_{\theta R}} S_1 dM \leq \int_{B_R} |\nabla f| dM \leq \int_{B_r \setminus B_{\theta R}} \frac{1}{(1-\theta)R} dM \leq \frac{1}{(1-\theta)R}\operatorname{vol}(B_R).$$

We observe that since M is a graph, if $\Omega_R = \{(x_1, .., x_{n+1}) \in \mathbb{R}^{n+1} | -R \leq x_{n+1} \leq R; \sqrt{x_1^2 + ... + x_n^2} \leq R\}$, then

$$\operatorname{vol}(B_R) \leq \int_{\Omega_R} 1 dx_1 ... dx_{n+1} = C(n) R^{n+1}.$$

Hence,

$$\int_{B_{\theta R}} S_1 dM \leq \frac{1}{(1-\theta)R}\operatorname{vol}(B_R) = \frac{C(n)}{1-\theta} R^n.$$

□

We have the following Corollary of Theorem 3.2

Corollary 4.1 *Any entire graph on $\mathbb{R}^n$ with nonnegative constant scalar curvature must have zero scalar curvature.*

Proof. Suppose by sake of contradiction that there exist a entire graph with $S_2 = const > 0$. Such graph is strongly stable and if $S_2 > 0$, we get that $S_1^2 = |A|^2 + 2S_2 > 0$, we obtain that S_1 does not change sign and we can choose the orientation in such way that $S_1 > 0$. Thus the graph has polynomial growth of the 1-volume. Thus we have a contradiction with Theorem 3.2. Thus it follows that $S_2 = 0$.

$\square$

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